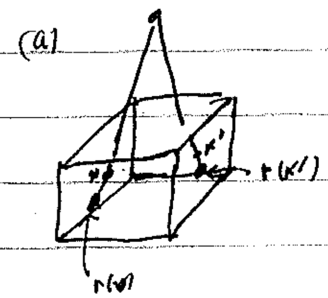
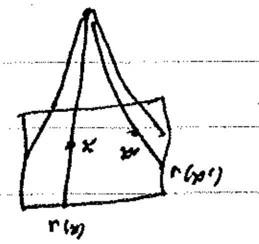
Homework 2 - Sketch of Solutions

#2 (a)



One dimension less, the picture is easier to draw



mapping $I \times I \to I \times 0 \cup \dot{I} \times I$

(b) Write $c_0 = c_{x_0}$. Define $l_0(t) = F(0,t)$, $l_1(t) = F(1,t)$

l_0, l_1 loops in A based at x_0. $l_0 \underset{G}{\simeq} c_0$, $l_1 \underset{H}{\simeq} c_0$

Define $\Phi' : I^2 \times 0 \cup \dot{I}^2 \times I \to X$ by

$\Phi'(x,t,0) = F(x,t)$ $\quad\quad$ $\Phi'(x,0,s) = f(x)$

$\Phi'(0,t,s) = G(t,s)$ $\quad\quad$ $\Phi'(x,1,s) = g(x)$

$\Phi(1,t,s) = H(t,s)$

Φ' is well-defined and continuous. Since $I^2 \times 0 \cup \dot{I}^2 \times I$ is a retract of I^3, Φ' extends to $\Phi : I^2 \times I \to X$ (set $\Phi = \Phi' r$, where r is the retraction). Then set

$$F'(x,t) = \Phi(x,t,1)$$

and so $f \underset{F'}{\simeq} g$

#3 Show $i_* \pi(A,x_0) \cap \operatorname{Ker} r_* = 0$ and $\pi(X,x_0) = \operatorname{Im} i_* + \operatorname{Ker} r_*$.

#4 The projection $p_1 : S^1 \times S^1 \to S^1 \times x_0$ is the retraction. It is not a dr because $S^1 \times S^1$ and $S^1 \times x_0 \approx S^1$ have different fundamental groups ($\mathbb{Z} \oplus \mathbb{Z}$ and $\mathbb{Z}$).

#5 $(id \cdot c)(x) = \mu(x, c(x)) = c_e(x)$ (mult. notation for $\pi(G,e)$)

$(id \cdot c)_* : \pi(G,e) \to \pi(G,e)$, $\alpha = [f] \in \pi(G,e)$

$f \cdot cf \sim (id \cdot c)(f) = c_e f = c'_e$, where $c'_e : I \to G$ is constant map

$\therefore \alpha \cdot c_*(\alpha) = 1$ so $c_*(\alpha) = \alpha^{-1}$. (Note: the proof for a topological group X that the two operations in $\pi(X)$ coincide holds for space G defined in Exercise 7.5)

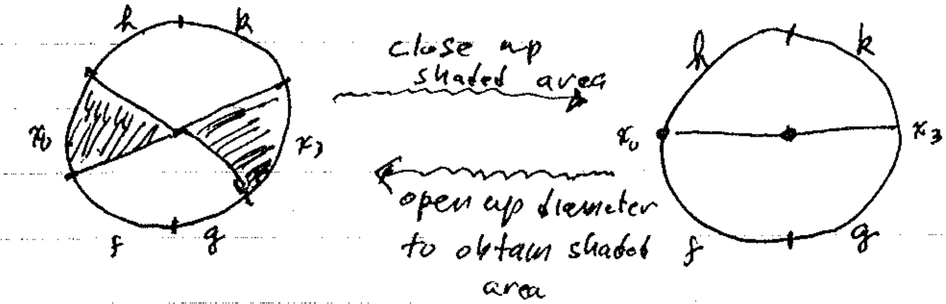
#6 Let $g \neq 1$ in G and $h \neq 1$ in H. Then $ghg^{-1}h^{-1}$ is a reduced word in $G * H$ and so $\neq 1$. $\therefore gh \neq hg$. Furthermore

$$gh,\ ghgh,\ ghgh, \ldots$$

are infinitely many distinct elements of $G*H$.

#8 Let $i: S^1 \to E^2$ be the inclusion and suppose $\tilde\varphi: E^2 \to X$ with $\tilde\varphi i = \varphi$. Since E^2 is contractible, $id \simeq c_0 : E^2 \to E^2$.
$\therefore \varphi = \tilde\varphi i = \tilde\varphi\, id\, i \simeq \tilde\varphi c_0 i = c_{\varphi(0)}$. $\therefore \varphi$ is nullhomotopic.
Conversely let $F: S^1 \times I \to X$, $F(x,0) = x_0$, $F(x,1) = \varphi x$.
Define $\tilde\varphi : E^2 \to X$ by $\tilde\varphi(tx) = F(x,t)$, $x \in S^1$, $t \in I$.

#9 Picture proof.



Other proof. Square 1 can be filled in $\iff c_{x_0} hk\, c_{x_3} \bar g \bar f \sim c_{x_0}$. Square 2 can be filled in $\iff hk \bar g \bar f \sim c_{x_0}$.

#11 $P \approx E^2/x \sim -x$ for $x \in S^1$. Let $E^2_+ \subseteq S^2$ be all $(x_1, x_2, x_3) \in S^2$ with $x_3 \geq 0$ (the upper cap). Then $E^2 \approx E^2_+$ (map $(x_1,x_2,x_3) \in E^2_+$ to (x_1, x_2)) $\therefore E^2/x \sim x \approx E^2_+/x \sim -x$ $x \in S^1$
Let $i: E^2_+ \to S^2$ be inclusion. Then i induces

$$i' : E^2_+/x \sim -x \to S^2/z \sim -z \qquad z \in S^2$$

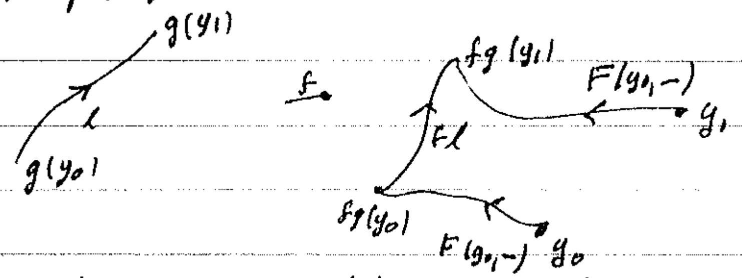
Show i' one-one, onto. i' cont. map from a compact space to a Hausdorff space which is a bijection. $\therefore i'$ is a homeo.

#12 Let X_0 be the path component containing x_0 and let $i: X_0 \to X$ be inclusion. Show $i_*: \pi(X_0, x_0) \to \pi(X, x_0)$ is isomorphism.

If f is a loop based at x_0, $f(I)$ is a p.c. space containing x_0. $\therefore f(I) \subseteq X_0$. $\therefore i_*$ onto. If f, g are loops in X_0 and they are equivalent in X with homotopy F, then $F(I \times I)$ is p.c containing x_0, so $F(I\times I) \subseteq X_0$. $\therefore i_*$ one-one.

#13 See the solution to Problem 13 Homework 1. We obtain $\mathbb{R}^2 - \{z_1, \dots, z_m\}$ where $m = 2n+1$. Use induction and the SvK theorem to prove that the fundamental group is the free group on $m = 2n+1$ generators

#14 Let $f : X \to Y$ be a homotopy equivalence with homotopy inverse $g : Y \to X$ and $\mathrm{id} \simeq fg$. Let $y_0, y_1 \in Y$ and let l be a path from $g(y_0)$ to $g(y_1)$. The following picture should enable you to finish the proof.



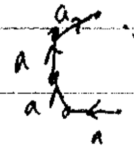
#15 Let U be an open nbh of y_0 homeo to the open n-ball with y_0 corresponding to the origin. Apply SvK to $U \cup M - y_0$ with intersection $U - y_0$. Since U is contractible

$$\pi(M) = \pi(M - y_0) / \overline{i_* \pi(U - y_0)}$$

If B is the open n-ball with center 0, then $B - 0 \simeq S^{n-1}$.

$\therefore \pi(U - y_0) \approx \pi(S^{n-1}) = 0$ since $n - 1 \geq 2$. $\therefore \pi(M) \approx \pi(M - y_0)$

#16 Take an m-gon P with all edges a identified as



Call the identification space P_m. Now if G is a f.g. abelian grp, then $G \cong \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{t} \oplus \mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_s}$

Let $X = \underbrace{S^1 \times \cdots \times S^1}_{t} \times P_{m_1} \times \cdots \times P_{m_s}$.